➤ The series in Example 1c is called an *alternating series* because the terms alternate in sign. Such series are discussed in detail in Section 8.6.

c. Writing out the first few terms of the series is helpful:

$$\sum_{k=2}^{\infty} 3(-0.75)^k = \underbrace{3(-0.75)^2}_{a} + \underbrace{3(-0.75)^3}_{ar} + \underbrace{3(-0.75)^4}_{ar^2} + \cdots.$$

We see that the first term of the series is $a = 3(-0.75)^2$, and the ratio of the series is $r = -0.75$. Because $|r| < 1$, the series converges, and its value is

$$\sum_{k=2}^{\infty} 3(-0.75)^k = \frac{3(-0.75)^2}{1-(-0.75)} = \frac{27}{28}.$$

Related Exercises 7–40

EXAMPLE 2 **Decimal expansions** Write $1.0\overline{35} = 1.0353535\ldots$ as a geometric serie and express its value as a fraction.

SOLUTION Notice that the decimal part of this number is a convergent geometric series with $a = 0.035$ and $r = 0.01$:

$$1.0353535\ldots = 1 + \underbrace{0.035 + 0.00035 + 0.0000035 + \cdots}_{\text{geometric series with } a\,=\,0.035 \text{ and } r\,=\,0.01}$$

Evaluating the series, we have

$$1.0353535\ldots = 1 + \frac{a}{1-r} = 1 + \frac{0.035}{1-0.01} = 1 + \frac{35}{990} = \frac{205}{198}.$$

Related Exercises 41–46

Telescoping Series

With geometric series, we carried out the entire evaluation process by finding a formu for the sequence of partial sums and evaluating the limit of the sequence. Not many in nite series can be subjected to this sort of analysis. With another class of series, calle **telescoping series**, it can be done. Here is an example.

EXAMPLE 3 **Telescoping series** Evaluate the following series.

a. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{3^k} - \frac{1}{3^{k+1}}\right)$ **b.** $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$

SOLUTION

a. The nth term of the sequence of partial sums is

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{3^k} - \frac{1}{3^{k+1}}\right) = \left(\frac{1}{3} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{3^3}\right) + \cdots + \left(\frac{1}{3^n} - \frac{1}{3^{n+1}}\right)$$

$$= \frac{1}{3} + \underbrace{\left(-\frac{1}{3^2} + \frac{1}{3^2}\right)}_{0} + \cdots + \underbrace{\left(-\frac{1}{3^n} + \frac{1}{3^n}\right)}_{0} - \frac{1}{3^{n+1}} \quad \text{Regroup terms.}$$

$$= \frac{1}{3} - \frac{1}{3^{n+1}} \quad \text{Simplify.}$$

➤ The series in Example 3a is also a geometric series and its value can be found using Theorem 8.7.

Observe that the interior terms of the sum cancel (or telescope) leaving a simple expression for S_n. Taking the limit, we find that

$$\sum_{k=1}^{\infty}\left(\frac{1}{3^k} - \frac{1}{3^{k+1}}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{3} - \underbrace{\frac{1}{3^{n+1}}}_{\to 0}\right) = \frac{1}{3}.$$

➤ See Section 7.4 for a review of partial fractions.

b. Using the method of partial fractions, the sequence of partial sums is

$$S_n = \sum_{k=1}^{n}\frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right).$$

Writing out this sum, we see that

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 + \underbrace{\left(-\frac{1}{2} + \frac{1}{2}\right)}_{0} + \underbrace{\left(-\frac{1}{3} + \frac{1}{3}\right)}_{0} + \cdots + \underbrace{\left(-\frac{1}{n} + \frac{1}{n}\right)}_{0} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}.$$

Again, the sum telescopes and all the interior terms cancel. The result is a simple formula for the nth term of the sequence of partial sums. The value of the series is

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1.$$

Related Exercises 47–58 ◄

SECTION 8.3 EXERCISES

Review Questions

1. What is the defining characteristic of a geometric series? Give an example.
2. What is the difference between a geometric sum and a geometric series?
3. What is meant by the *ratio* of a geometric series?
4. Does a geometric sum always have a finite value?
5. Does a geometric series always have a finite value?
6. What is the condition for convergence of the geometric series $\displaystyle\sum_{k=0}^{\infty} ar^k$?

Basic Skills

7–18. Geometric sums *Evaluate the following geometric sums.*

7. $\displaystyle\sum_{k=0}^{8} 3^k$
8. $\displaystyle\sum_{k=0}^{10}\left(\frac{1}{4}\right)^k$
9. $\displaystyle\sum_{k=0}^{20}\left(\frac{2}{5}\right)^{2k}$
10. $\displaystyle\sum_{k=4}^{12} 2^k$
11. $\displaystyle\sum_{k=0}^{9}\left(-\frac{3}{4}\right)^k$
12. $\displaystyle\sum_{k=1}^{5}(-2.5)^k$
13. $\displaystyle\sum_{k=0}^{6}\pi^k$
14. $\displaystyle\sum_{k=1}^{10}\left(\frac{4}{7}\right)^k$
15. $\displaystyle\sum_{k=0}^{20}(-1)^k$
16. $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27}$
17. $\frac{1}{4} + \frac{1}{12} + \frac{1}{36} + \frac{1}{108} + \cdots + \frac{1}{2916}$
18. $\frac{1}{3} + \frac{1}{5} + \frac{3}{25} + \frac{9}{125} + \cdots + \frac{243}{15{,}625}$

19–34. Geometric series *Evaluate the geometric series or state that it diverges.*

19. $\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k$
20. $\displaystyle\sum_{k=0}^{\infty}\left(\frac{3}{5}\right)^k$
21. $\displaystyle\sum_{k=0}^{\infty} 0.9^k$
22. $\displaystyle\sum_{k=0}^{\infty}\frac{2^k}{7^k}$
23. $\displaystyle\sum_{k=0}^{\infty} 1.01^k$
24. $\displaystyle\sum_{j=0}^{\infty}\left(\frac{1}{\pi}\right)^j$
25. $\displaystyle\sum_{k=1}^{\infty} e^{-2k}$
26. $\displaystyle\sum_{m=2}^{\infty}\frac{5}{2^m}$
27. $\displaystyle\sum_{k=1}^{\infty} 2^{-3k}$
28. $\displaystyle\sum_{k=3}^{\infty}\frac{3\cdot 4^k}{7^k}$
29. $\displaystyle\sum_{k=4}^{\infty}\frac{1}{5^k}$
30. $\displaystyle\sum_{k=0}^{\infty}\left(\frac{4}{3}\right)^{-k}$
31. $\displaystyle\sum_{k=0}^{\infty}\left(\frac{e}{\pi}\right)^k$
32. $\displaystyle\sum_{k=1}^{\infty}\frac{3^{k-1}}{4^{k+1}}$
33. $\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k 5^{6-k}$
34. $\displaystyle\sum_{k=2}^{\infty}\left(\frac{3}{8}\right)^{3k}$

35–40. Geometric series with alternating signs *Evaluate the geometric series or state that it diverges.*

35. $\displaystyle\sum_{k=0}^{\infty}\left(-\frac{9}{10}\right)^k$
36. $\displaystyle\sum_{k=1}^{\infty}\left(-\frac{2}{3}\right)^k$
37. $\displaystyle 3\sum_{k=0}^{\infty}\frac{(-1)^k}{\pi^k}$
38. $\displaystyle\sum_{k=1}^{\infty}(-e)^{-k}$
39. $\displaystyle\sum_{k=2}^{\infty}(-0.15)^k$
40. $\displaystyle\sum_{k=1}^{\infty} 3\left(-\frac{1}{8}\right)^{3k}$

41–46. Decimal expansions *Write each repeating decimal first as a geometric series, then as a fraction (a ratio of two integers).*

41. 0.121212... **42.** 1.252525... **43.** 0.456456...

44. 1.00393939... **45.** 0.00952952... **46.** 5.12838383...

47–58. Telescoping series *For the following telescoping series, find a formula for the nth term of the sequence of partial sums $\{S_n\}$. Then evaluate $\lim_{n\to\infty} S_n$ to obtain the value of the series or state that the series diverges.*

47. $\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)$ **48.** $\sum_{k=1}^{\infty}\left(\frac{1}{k+2}-\frac{1}{k+3}\right)$

49. $\sum_{k=1}^{\infty}\frac{1}{(k+1)(k+2)}$ **50.** $\sum_{k=0}^{\infty}\frac{1}{(3k+1)(3k+4)}$

51. $\sum_{k=1}^{\infty}\ln\left(\frac{k+1}{k}\right)$ **52.** $\sum_{k=1}^{\infty}\left(\sqrt{k+1}-\sqrt{k}\right)$

53. $\sum_{k=1}^{\infty}\frac{1}{(k+p)(k+p+1)}$, where p is a positive integer

54. $\sum_{k=1}^{\infty}\frac{1}{(ak+1)(ak+a+1)}$, where a is a positive integer

55. $\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k+1}}-\frac{1}{\sqrt{k+3}}\right)$

56. $\sum_{k=0}^{\infty}\left[\sin\left(\frac{(k+1)\pi}{2k+1}\right)-\sin\left(\frac{k\pi}{2k-1}\right)\right]$

57. $\sum_{k=0}^{\infty}\frac{1}{16k^2+8k-3}$ **58.** $\sum_{k=1}^{\infty}[\tan^{-1}(k+1)-\tan^{-1}k]$

Further Explorations

59. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. $\sum_{k=1}^{\infty}\left(\frac{\pi}{e}\right)^{-k}$ is a convergent geometric series.

b. If a is a real number and $\sum_{k=12}^{\infty}a^k$ converges, then $\sum_{k=1}^{\infty}a^k$ converges.

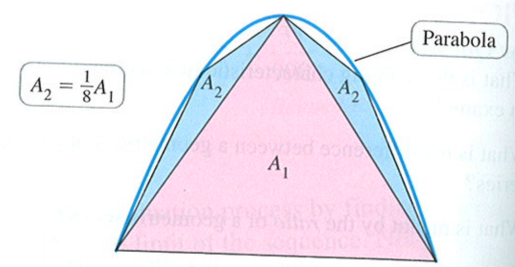
c. If the series $\sum_{k=1}^{\infty}a^k$ converges and $|a|<|b|$, then the series $\sum_{k=1}^{\infty}b^k$ converges.

60. Zeno's paradox The Greek philosopher Zeno of Elea (who lived about 450 B.C.) invented many paradoxes, the most famous of which tells of a race between the swift warrior Achilles and a tortoise. Zeno argued

The slower when running will never be overtaken by the quicker; for that which is pursuing must first reach the point from which that which is fleeing started, so that the slower must necessarily always be some distance ahead.

In other words, giving the tortoise a head start, Achilles will never overtake the tortoise because every time Achilles reaches the point where the tortoise was, the tortoise has moved ahead. Resolve this paradox by assuming that Achilles gives the tortoise a 1-mi head st[art] and runs 5 mi/hr to the tortoise's 1 mi/hr. How far does Achilles ru[n] before he overtakes the tortoise, and how long does it take?

61. Archimedes' quadrature of the parabola The Greeks solved several calculus problems almost 2000 years before the discove[ry] of calculus. One example is Archimedes' calculation of the area of the region R bounded by a segment of a parabola, which he did using the "method of exhaustion." As shown in the figure, the id[ea] was to fill R with an infinite sequence of triangles. Archimedes began with one triangle inscribed in the parabola, with area A_1, and proceeded in stages, with the number of new triangles doubling at each stage. He was able to show (the key to the solution) that at each stage, the area of a new triangle is $\frac{1}{8}$ of the area of a triangle at the previous stage; for example, $A_2=\frac{1}{8}A_1$, and so forth. Show, as Archimedes did, that the area of R is $\frac{4}{3}$ times the area of A_1.



62. Value of a series

a. Find the value of the series

$$\sum_{k=1}^{\infty}\frac{3^k}{(3^{k+1}-1)(3^k-1)}.$$

b. For what value of a does the series

$$\sum_{k=1}^{\infty}\frac{a^k}{(a^{k+1}-1)(a^k-1)}$$

converge, and in those cases, what is its value?

Applications

T **63. House loan** Suppose you take out a home mortgage for \$180,000 at a monthly interest rate of 0.5%. If you make payments of \$1000 per month, after how many months will the loan balance be zero? Estimate the answer by graphing the sequence of loan balances and then obtain an exact answer.

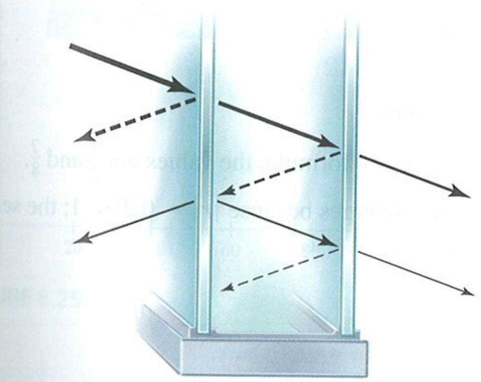
T **64. Car loan** Suppose you borrow \$20,000 for a new car at a monthly interest rate of 0.75%. If you make payments of \$600 per month, after how many months will the loan balance be zero? Estimate the answer by graphing the sequence of loan balances and then obtain an exact answer.

65. Fish harvesting A fishery manager knows that her fish population naturally increases at a rate of 1.5% per month. At the end of each month, 120 fish are harvested. Let F_n be the fish population after the nth month, where $F_0 = 4000$ fish. Assuming that this process continues indefinitely, what is the long-term (steady-state) population of the fish?

66. Periodic doses Suppose that you take 200 mg of an antibiotic every 6 hr. The half-life of the drug is 6 hr (the time it takes for half of the drug to be eliminated from your blood). If you continue this regimen indefinitely, what is the long-term (steady-state) amount of antibiotic in your blood?

67. China's one-son policy In 1978, in an effort to reduce population growth, China instituted a policy that allows only one child per family. One unintended consequence has been that, because of a cultural bias toward sons, China now has many more young boys than girls. To solve this problem, some people have suggested replacing the one-child policy with a one-son policy: A family may have children until a boy is born. Suppose that the one-son policy were implemented and that natural birth rates remained the same (half boys and half girls). Using geometric series, compare the total number of children under the two policies.

68. Double glass An insulated window consists of two parallel panes of glass with a small spacing between them. Suppose that each pane reflects a fraction p of the incoming light and transmits the remaining light. Considering all reflections of light between the panes, what fraction of the incoming light is ultimately transmitted by the window? Assume the amount of incoming light is 1.



69. Bouncing ball for time Suppose a rubber ball, when dropped from a given height, returns to a fraction p of that height. How long does it take for a ball dropped from 10 m to come to rest? In the absence of air resistance, a ball dropped from a height h requires $\sqrt{2h/g}$ seconds to fall to the ground, where $g\approx 9.8\text{ m/s}^2$ is the acceleration due to gravity. The time taken to bounce *up* to a given height equals the time to fall from that height to the ground.

70. Multiplier effect Imagine that the government of a small community decides to give a total of \$$W$, distributed equally, to all of its citizens. Suppose that each month each citizen saves a fraction p of his or her new wealth and spends the remaining $1-p$ in the community. Assume no money leaves or enters the community, and all of the spent money is redistributed throughout the community.

a. If this cycle of saving and spending continues for many months, how much money is ultimately spent? Specifically, by what factor is the initial investment of \$$W$ increased? (Economists refer to this increase in the investment as the multiplier effect.)

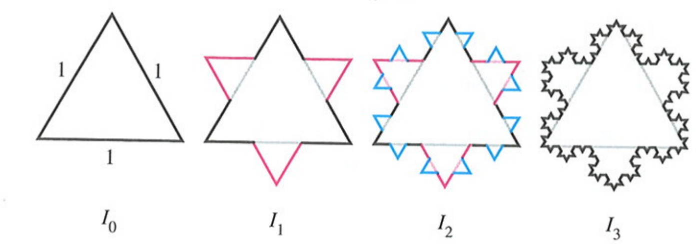
b. Evaluate the limits $p\to 0$ and $p\to 1$ and interpret their meanings.

(See Guided Projects for more on economic stimulus packages.)

71. Snowflake island fractal The fractal called the *snowflake island* (or *Koch island*) is constructed as follows: Let I_0 be an equilateral triangle with sides of length 1. The figure I_1 is obtained by replacing the middle third of each side of I_0 by a new outward equilateral triangle with sides of length 1/3 (see figure). The process is repeated where I_{n+1} is obtained by replacing the middle third of each side of I_n by a new outward equilateral triangle with sides of length $1/3^{n+1}$. The limiting figure as $n\to\infty$ is called the snowflake island.

a. Let L_n be the perimeter of I_n. Show that $\lim_{n\to\infty}L_n=\infty$.

b. Let A_n be the area of I_n. Find $\lim_{n\to\infty}A_n$. It exists!



Additional Exercises

72. Decimal expansions

a. Consider the number 0.555555..., which can be viewed as the series $5\sum_{k=1}^{\infty}10^{-k}$. Evaluate the geometric series to obtain a rational value of 0.555555...

b. Consider the number 0.54545454..., which can be represented by the series $54\sum_{k=1}^{\infty}10^{-2k}$. Evaluate the geometric series to obtain a rational value of the number.

c. Now generalize parts (a) and (b). Suppose you are given a number with a decimal expansion that repeats in cycles of length p, say, $n_1, n_2 \ldots n_p$, where $n_1,\ldots,n_p$ are integers between 0 and 9. Explain how to use geometric series to obtain a rational form of the number.

d. Try the method of part (c) on the number 0.123456789123456789....

e. Prove that $0.\overline{9}=1$.

73. Remainder term Consider the geometric series $S=\sum_{k=0}^{\infty}r^k$, which has the value $1/(1-r)$ provided $|r|<1$, and let $S_n=\sum_{k=0}^{n-1}r^k=\frac{1-r^n}{1-r}$ be the sum of the first n terms. The remainder R_n is the error in approximating S by S_n. Show that

$$R_n=|S-S_n|=\left|\frac{r^n}{1-r}\right|.$$

74–77. Comparing remainder terms *Use Exercise 73 to determine how many terms of each series are needed so that the partial sum is within 10^{-6} of the value of the series (that is, to ensure $R_n < 10^{-6}$).*

74. **a.** $\sum_{k=0}^{\infty} 0.6^k$ **b.** $\sum_{k=0}^{\infty} 0.15^k$ **75.** **a.** $\sum_{k=0}^{\infty} (-0.8)^k$ **b.** $\sum_{k=0}^{\infty} 0.2^k$

76. **a.** $\sum_{k=0}^{\infty} 0.72^k$ **b.** $\sum_{k=0}^{\infty} (-0.25)^k$ **77.** **a.** $\sum_{k=0}^{\infty} \left(\frac{1}{\pi}\right)^k$ **b.** $\sum_{k=0}^{\infty} \left(\frac{1}{e}\right)^k$

78. Functions defined as series Suppose a function f is defined by the geometric series $f(x) = \sum_{k=0}^{\infty} x^k$.

a. Evaluate $f(0), f(0.2), f(0.5), f(1)$, and $f(1.5)$, if possible.
b. What is the domain of f?

79. Functions defined as series Suppose a function f is defined by the geometric series $f(x) = \sum_{k=0}^{\infty} (-1)^k x^k$.

a. Evaluate $f(0), f(0.2), f(0.5), f(1)$, and $f(1.5)$.
b. What is the domain of f?

80. Functions defined as series Suppose a function f is defined by the geometric series $f(x) = \sum_{k=0}^{\infty} x^{2k}$.

a. Evaluate $f(0), f(0.2), f(0.5), f(1)$, and $f(1.5)$.
b. What is the domain of f?

81. Series in an equation For what values of x does the geometric series

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{1}{1+x}\right)^k$$

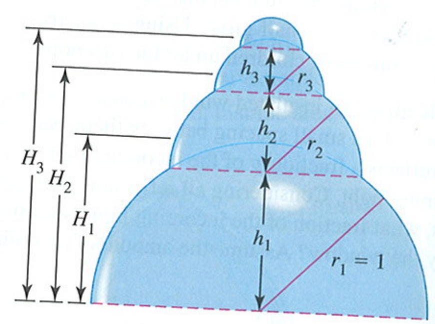
converge? Solve $f(x) = 3$.

82. Bubbles Imagine a stack of hemispherical soap bubbles with decreasing radii $r_1 = 1, r_2, r_3, \ldots$ (see figure). Let h_n be the distance between the diameters of bubble n and bubble $n+1$, and let H_n be the total height of the stack with n bubbles.

a. Use the Pythagorean theorem to show that in a stack with n bubbles, $h_1^2 = r_1^2 - r_2^2$, $h_2^2 = r_2^2 - r_3^2$, and so forth. Note that $h_n = r_n$.
b. Use part (a) to show that the height of a stack with n bubbles is

$$H_n = \sqrt{r_1^2 - r_2^2} + \sqrt{r_2^2 - r_3^2} + \cdots + \sqrt{r_{n-1}^2 - r_n^2} + r_n.$$

c. The height of a stack of bubbles depends on how the radii decrease. Suppose that $r_1 = 1, r_2 = a, r_3 = a^2, \ldots, r_n = a^n$, where $0 < a < 1$ is a fixed real number. In terms of a, find the height H_n of a stack with n bubbles.
d. Suppose the stack in part (c) is extended indefinitely ($n \to \infty$). In terms of a, how high would the stack be?
e. Challenge problem: Fix n and determine the sequence of radii $r_1, r_2, r_3, \ldots, r_n$ that maximizes H_n, the height of the stack with n bubbles.



QUICK CHECK ANSWERS

1. b and c **2.** Using the formula, the values are $\frac{3}{2}$ and $\frac{7}{8}$. **3.** 1 **4.** The first converges because $|r| = 0.2 < 1$; the second diverges because $|r| = 2 > 1$.

8.4 The Divergence and Integral Tests

With geometric series and telescoping series, the sequence of partial sums can be found and its limit can be evaluated (when it exists). Unfortunately, it is difficult or impossible to find an explicit formula for the sequence of partial sums for most infinite series. Therefore, it is difficult to obtain the exact value of most convergent series.

In this section, we explore methods to determine whether or not a given infinite series converges, which is simply a *yes/no* question. If the answer is *no*, the series diverges, and there are no more questions to ask. If the answer is *yes*, the series converges and it may be possible to estimate its value.

The Harmonic Series

We begin with an example that has a surprising result. Consider the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots,$$

a famous series known as the **harmonic series**. Does it converge? Suppose you try to answer this question by writing out the terms of the sequence of partial sums:

We analyze S_n numerically because an explicit formula for S_n does not exist.

$$S_1 = 1 \qquad S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \qquad S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$

$$\vdots$$

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

$$\vdots$$

Have a look at the first 200 terms of the sequence of partial sums shown in Figure 8.25. What do you think—does the series converge? The terms of the sequence of partial sums increase, but at a decreasing rate. They could approach a limit or they could increase without bound.

Computing additional terms of the sequence of partial sums does not provide conclusive evidence. Table 8.3 shows that the sum of the first million terms is less than 15; the sum of the first 10^{40} terms—an unimaginably large number of terms—is less than 100. This is a case in which computation alone is not sufficient to determine whether a series converges. We return to this example later with more refined methods.

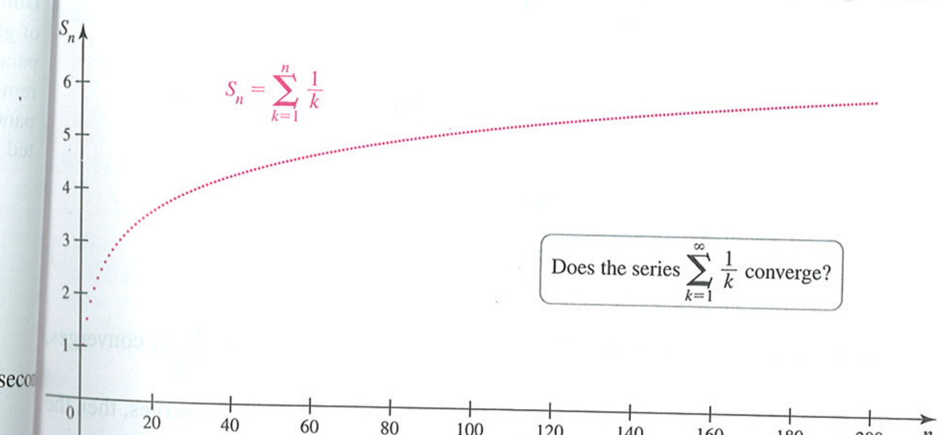


FIGURE 8.25

Table 8.3

n	S_n	n	S_n
10^3	≈ 7.49	10^{10}	≈ 23.60
10^4	≈ 9.79	10^{20}	≈ 46.63
10^5	≈ 12.09	10^{30}	≈ 69.65
10^6	≈ 14.39	10^{40}	≈ 92.68

Properties of Convergent Series

For now, we restrict our attention to series with positive terms; that is, series of the form $\sum a_k$, where $a_k > 0$. The notation $\sum a_k$, without initial and final values of k, is used to refer to a general infinite series.

➤ The **leading terms** of an infinite series are those at the beginning with a small index. The **tail** of an infinite series consists of the terms at the "end" of the series with a large and increasing index. The convergence or divergence of an infinite series depends on the tail of the series, while the value of a convergent series is determined primarily by the leading terms.

THEOREM 8.8 Properties of Convergent Series

1. Suppose $\sum a_k$ converges to A and let c be a real number. The series $\sum ca_k$ converges and $\sum ca_k = c\sum a_k = cA$.
2. Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum (a_k \pm b_k)$ converges and $\sum (a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$.
3. *Whether* a series converges does not depend on a finite number of terms added to or removed from the series. Specifically, if M is a positive integer, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ both converge or both diverge. However, the *value* of a convergent series does change if nonzero terms are added or deleted.